Notes on Tuning

Synthesis requires some calculation to convert the note number coming in to a frequency for the oscillator. In the synthesis tutorials, this is provided by the mtof object, which converts according to equal temperament. There are many other ways to do this, and some produce very interesting results.

A little digression on the history of temperaments

Temperament refers to the relative tuning of the notes in a scale. It's not the alternation of half and whole steps that distinguish modes, but the size of the steps that we are concerned with.

The math begins with the consideration of octaves. Each octave is twice the pitch of the lower so notes some number of octaves apart are related by a power of 2 as related in formula 1.

$$F = F_r * 2^n$$

Formula 1.

F_r is a reference frequency, and n is the number of octaves. Incidentally, n can be a negative number indicating an octave down[1]. The octave can be divided into smaller intervals a number of ways.

Circle of Fifths

Pythagoras discovered that the interval of a fifth can be found by multiplying the low frequency by 3/2. (He was actually measuring string lengths, but frequency is directly related to string length if all else is equal.) The major second can be found by going up another fifth and down an octave, and so on to produce some notes the Greeks found interesting. You may expect this the process to eventually return to an octave form the starting pitch, but as table 1 shows, that is not possible:

ops	0	1	2	3	4	5	6	7	8	9	10	11	12
P C	0	7	2	9	4	11	6	1	8	3	10	5	0
freq	440	660	495	742	557	835	626	470	705	529	792	595	892

Table 1.

The first row shows the number of times the base frequency is multiplied by 3/5. The second row shows the pitch class of the result, and the third row shows the resulting

[1] A reminder about exponents:
2^2 means 2 x 2
2^3 means 2 x 2 x 2
2^0 means 1
2^{-2} means 2 / 2
2^{-3} means 2 / 2 /2

frequency[2]. The final step would be expected to produce 880 Hz, but is sharp by 12 Hz. This error is the Pythagorean or ditonic comma. The scale can be made workable by hiding the bad fifth in a little used position. For instance, if a D scale is constructed by going down a fourth (4:3) five times and up seven, the comma occurs at D#.

Just Intonation

This system worked for two millennia, but more problems arose when triadic harmony came into style around the fifteenth century. Minstrels and folk singers used thirds in popular songs, but when these harmonies were attempted on the church organs, they did not work. The major thirds found in the Pythagorean scale are rather sharp compared to the nice consonance produced by a ratio of 5:4. Likewise, the minor thirds are a bit flat, compared to those built on the ratio 6:5. (These ratios are first mentioned in the writings of Ptolemy in the second century, but were considered less pure than 3:2, and should not be used in God's hearing. Speculation and commentary about the celestial and spiritual meanings of simple ratios was a favorite topic of philosophers from antiquity.)

To accommodate the new harmonies, a combination scale called just intonation was developed. Just intonation is produced by moving by combinations of fifths, fourths and major thirds.

P C	0	2	4	5	7	9	11	12
Interval	0	P5 -P4	M3	P4	P5	P4+M3	P5+M3	8va
Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1
Freq	440	495	550	587	660	733	825	880

Table 2.

Table 2 shows the construction of a just diatonic scale. No interval requires more than two operations, so there is little cumulative error. The fifths and thirds of the major chords are very nice. There are problems with the minor chords however. The interval m3 from pitch class 4 to 7 is dead on 6:5, but the interval from 2 to the 5 is severely flat. The fifth from pitch class 9 to 4 is good, but the interval 2 to 9 is far from perfect. In fact it is named "the wolf" The upshot is that various chords have very different flavors, many are beat free and beautiful, but others are unusable. When the system is extended to include the black keys, we discover that modulation produces even stranger results.

Mean Tone

Of course this is only theory, or was in the days before frequency counters.[3] What instrument builders actually did was construct harps and organ pipes according to these general ideas, then fudge the tuning by loosening the strings and widening the pipes. The accuracy of the tuning is determined by listening for beats in pitches that should be a fifth or an octave apart. To produce an instrument that would play thirds well, Renaissance instrument makers learned to tune by the Pythagorean method, but flatten each fifth to the

[2] This sort of chart is commonly shown as a circle, with the scale around the circumference and the fifths pattern drawn as a twelve pointed star.

[3] Or even the concept of frequency, which was not observed until the 18th century.

point of three beats per second. This so called "mean tone" method produced fifths only slightly flat and lovely thirds and sixths. When extended into the black keys, minor modes sounded beautiful and even some limited modulation was possible, but composers still had to avoid the wolf. In practice, organs were only playable in a few keys, harpsichords needed to be tuned for the programmed composition, and wind instruments sported different keys for D sharp and E flat.

Well Temperament

A fifth that is changed from the pure ratio of 3:2 is called "tempered". Theorists describing tuning systems refer to the amount of tempering in fractions of the Pythagorean comma.[4] During the Baroque period, dozens (if not hundreds) of tuning schemes were published- one of the most successful was by Andreas Werckmeister, who called his systems (he proposed many[5]) "well temperament". It is generally assumed that this is the tuning Bach had in mind for the WTC. Another well scale often encountered is Vallotti-Young[6], a bit closer to equal temperament than Werckmeister.

Equal Temperament

The notion of spreading the comma evenly among all of the fifths was first championed in print[7] by Vincenzo Galilei (Galileo's father) in 1582. There was a lot of resistance to this idea, first from philosophers still hung up on the Greek's sacred ratios, and then by scientists studying string modes and sympathetic vibration. Eventually, the sheer practicality of equal temperament has made it the predominant tuning in western music. Jean-Philippe Rameau should probably get the credit, because his writings in the first half of the 18th century not only encouraged equal temperament, they popularized harmonic practices that made ET practically obligatory.

In modern terms equal temperament is based on 12 steps in the octave. Formula 1 is easily modified to produce formula 2:

$$F = F_r * 2^{n/12}$$

Formula 2.

Now n refers to the number of semitones in an interval. The ratio of a semitone is the twelfth root of two, a number that would have given Pythagoras dyspepsia. An approximation of the twelfth root of two is 1.059463....

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[4] Or sometimes the error encountered at the third, called the syntonic comma. They are nearly the same.
[5] Some included combinations of raised as well as lowered fifths.
[6] Invented independently by two people rather late in the game. Young is nearly a transposition of Vallotti, and the two are usually considered the same.
[7] Lutemakers had actually been doing it for a long time. If you want to make a fretted string instrument, that's what you have to do.

Instead of talking about ratios and commas to describe interval tuning, we now use cents, defined as hundredths of a semitone. When an interval is expressed in cents the formula becomes:

$$F = F_r * 2^{c/1200}$$

Formula 3.
Here are the intervals in cents[8] of some scales:

PC	0	1	2	3	4	5	6	7	8	9	10	11	12
ET	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Py	0	90	204	294	408	498	611	702	792	906	996	1109	1214
JI	0	112	182	316	386	498	590	702	814	884	1018	1088	1200
MT	0	76	193	310	386	503	579	697	773	890	1007	1083	1200
W	0	90	192	294	390	498	588	696	792	888	996	1092	1200
VY	0	94	195	298	392	500	592	698	796	894	1000	1092	1200

Table 3.

Mean tone and well temperaments are pretty much limited to historical recreations these days, but just intonation has not gone away. There are composers who believe the timbral advantages of rationally tuned intervals are worth going to the effort of building instruments to play them. Lou Harrison and Harry Partch were famous champions of just and extended rational systems, and many contemporary composers continue the work.

The equal temperament story does not end with Rameau, either. If you can divide the octave into 12 parts, why not 24 or 41? Forty one notes (obviously quite close together) give you the ability to write chords as pure sounding as just followed by the wolf fifth on the same root. Charles Ives wrote piano duets, specifying that one piano be tuned a quarter tone flat. There are a lot of interesting sonorities available in such systems[9].

Equal temperament need not be based on the octave, for that matter. Bell labs computer scientist John Pierce designed scales based on dividing the 12th (the second mode of vibration for clarinets and similar instruments) various ways. The Bohlen-Pierce scale of 13 steps is becoming popular enough for builders to create physical instruments to play it.

The flexibility of computer systems has eliminated the problems of how to arrange keys or where to put frets. It's even possible to use artificial intelligence techniques to adjust the tuning to the musical situation to provide the ever changing relationships vocalists have always used.

[8] To do cents, you need the 1200th root of two, which is 1.000577789.
[9] And of course many non-western scales are equal tempered as well. For instance, modern Arabic scales are based on a 24 tone system.

Playing the scales

With a set of tunings in cents, it is easy to construct an MSP patch to play different scales. The heart of the patch will be a poly~[10] subpatch:

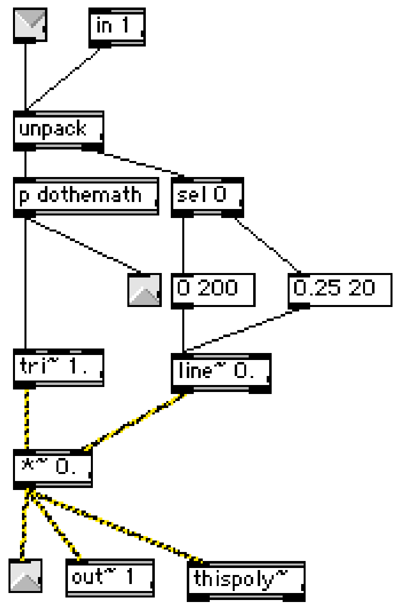

Figure 1.

This produces a rather dull triangle wave, but it can easily be elaborated. Triangles are good at bringing out differences in pitch. The work of calculating a frequency from a note number is in the dothemath subpatcher.

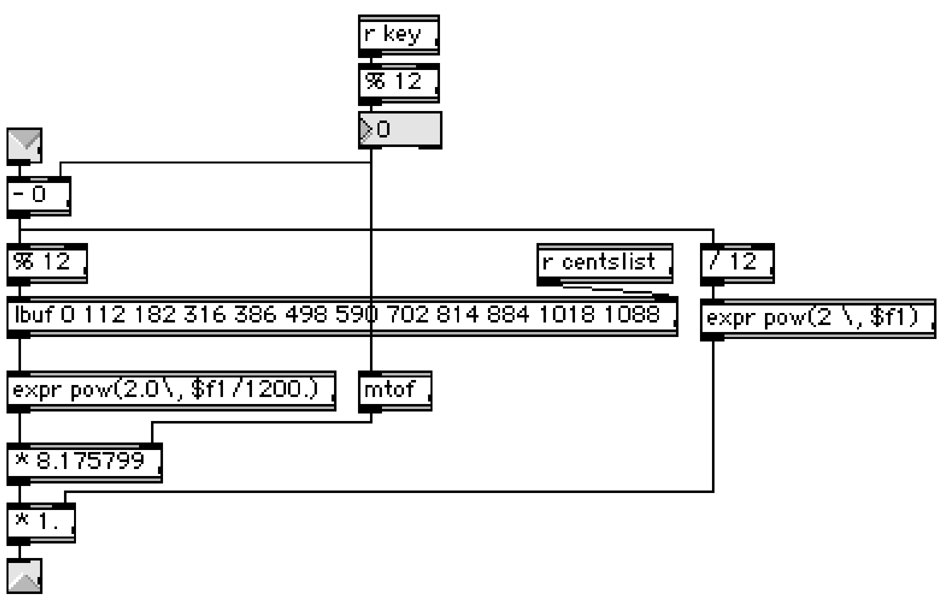

Figure 2.

[10] If you aren't familiar with poly~, look at my tutorial "Working with Poly~".

When a note arrives it is used to look up the cents interval, which is fed into an expr with the power of 2 part of formula 3. The result is multiplied by a base frequency to produce the desired value. The key message will do two things. It will transpose the note down and raise the base to give the desired pitch. In other words, the interval pattern will be transposed, but the notes will not.

The final step is a multiplication by a power of two to produce the proper octave, as in formula 1.

These are embedded in a master patcher.

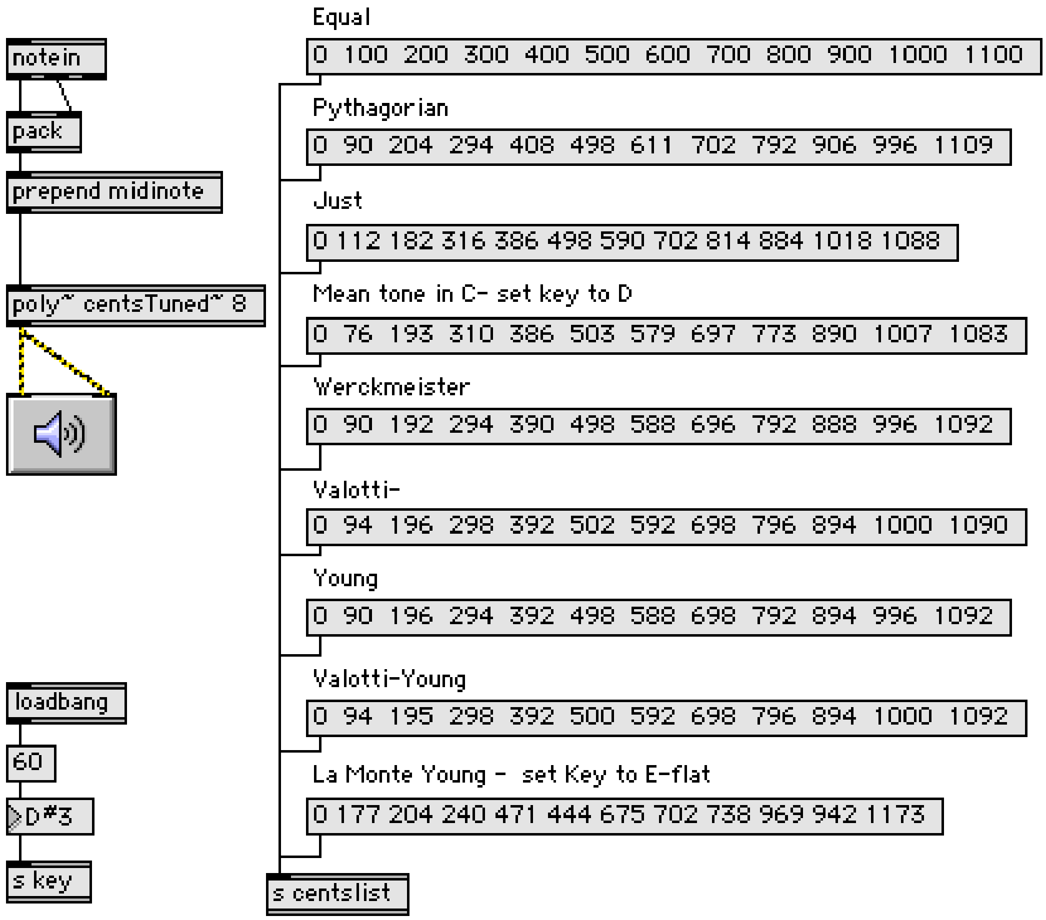

Figure 3.

Figure 3 combines the mechanism for playing the notes with some lists of different tunings (It would probably be neater in a coll, but this form makes it easy to see the action.) To retune the instrument, simply click on a list.

Other Equal Temperaments

A simple modification of these patchers will allow us to explore microtones.

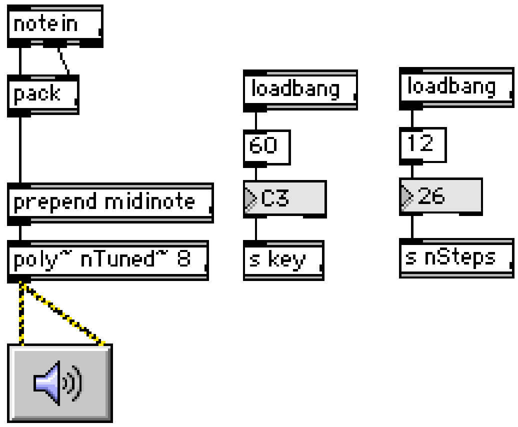

Figure 4.
Figure 4 is a master for playing a poly sub patch called nTuned, which is exactly like figure 1, except the dothemath subpatcher is now:

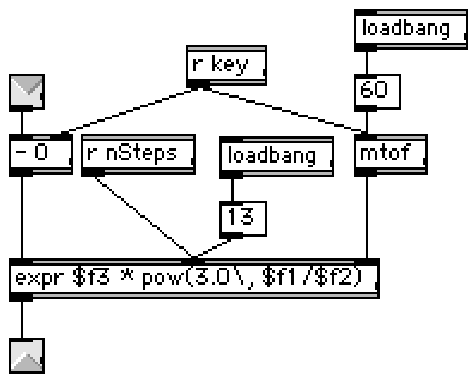

Figure 5.
Here we've taken formula 2 a step farther:

$$F = F_r * 2^{n/s}$$

Formula 4.
By adding another variable s for the number of steps in the octave, we can have as many tones as we like. In figure 5 n is the number of steps away from note 60 (or whatever is specified by key). The value from nSteps sets s.

One more change to dothemath will generate Bohlen-Pierce scales:

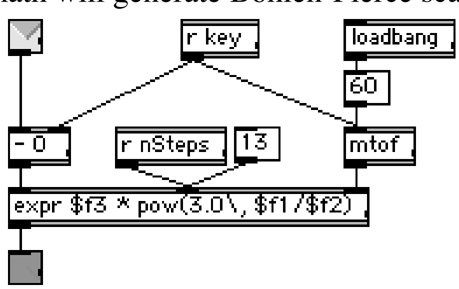

Figure 6.

Algorithmic tuning

There's one more refinement I like to add to the playCents patcher. None of the keyboard tunings really captures the flexibility and sonority that most ensembles, especially vocalists, provide. The whole business of fudging one interval to make another sound better is really a response to the problems of mechanical keyboards. Why not tune every note every time? One approach that works well is to set playCents to just intonation, and retune according to the current root. My tutorial Max & Chords shows how to analyze MIDI data to find a root. Here's a patcher that does that and sends the result to the patch that is playing. This version plays melodies in equal temperament, but harmonizes in just based on the root of the current chord.

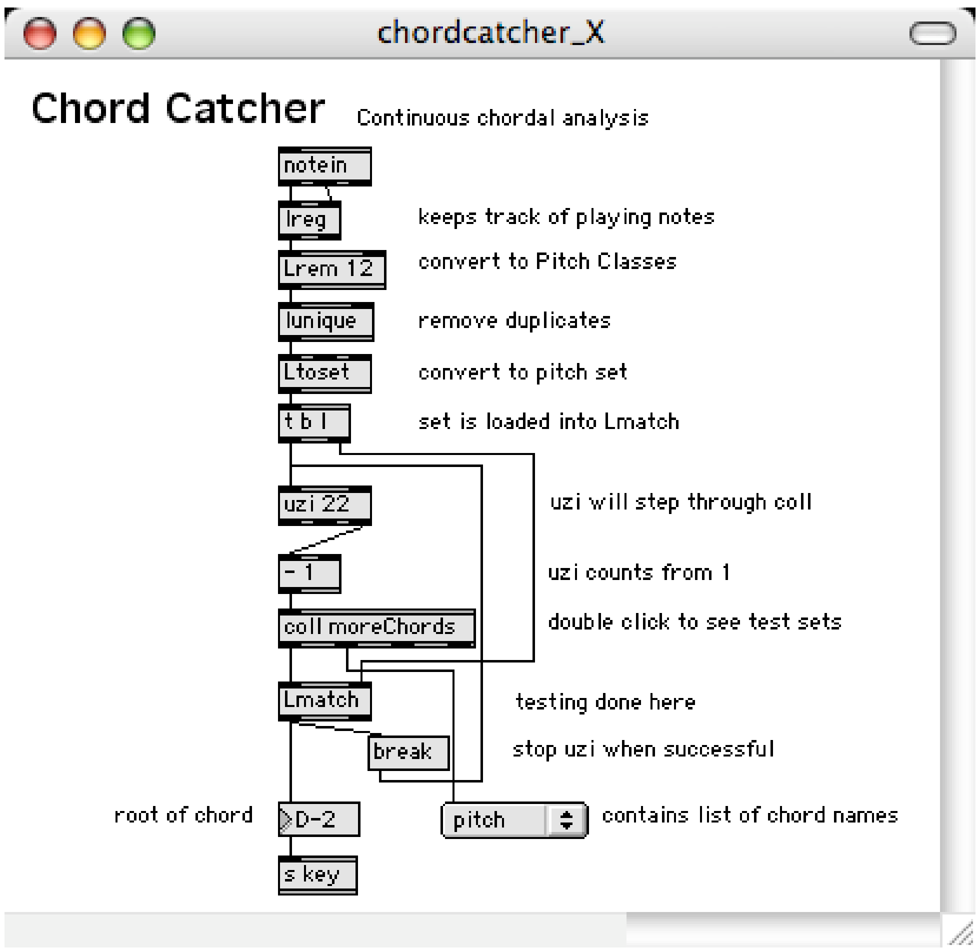

Figure 7.

For details of how this works and the contents of the moreChords coll, see the essay Max & Chords.