

Befriending Math

Almost all of us have a math limit. That's the point where we lose confidence in our mastery of numbers, and feel there's no way to understand the expressions we see in front of us. This paper has advice on how to expand your math horizons.

There are a lot of reasons our math skills may be limited.:

About 1 in 20 have dyscalculia, a real inability to mentally manipulate digits. It can go with the more familiar dyslexia, or be independent. Aside from "not getting math", dyscalculia usually includes a poor concept of the passage of time, frequently getting lost, and an inability to count backwards. Dyscalculia is a result of brain function, and may not affect other parts of your life. It can be overcome. Many famous scientists, including Albert Einstein were dyscalculic. It turns out that large regions of math don't require numbers at all. There are a lot of suggestions at dyscalculia.org. The site is primarily aimed at teaching students with dyscalculia, but since we are talking about self-study here, it might be appropriate. I was surprised to find that one of my favorite techniques, working on graph paper, is on their list of good practices.

Many more have math anxiety- a fear of working with numbers. This manifests as physical symptoms when confronted by math problems. Pounding heart, sweating, an intense desire to hide or flee. A common cause of math anxiety is bad teaching. Students who have been rushed in their studies, then humiliated because they haven't kept up often develop anxiety. Teachers (especially math teachers) forget that no one is born knowing math. Doing math is as difficult to learn as playing the guitar or speaking a foreign language. It takes time and steady directed practice. Anxiety is common among students who have had to change schools a lot or at a crucial time. It doesn't help that math teaching is subject to fads, including some methodologies that just don't work. A Google search on "math anxiety tips" pulled about 450,000 hits. Most of these duplicate the same advice, which center on de-stressing and practice.

Most of us just find math boring. Even if we do well in math class, there is no reward for practicing after the class is finished, let alone into adult life. We forget most math beyond figuring out the price of "buy two, get one free", and anything required in our daily job. Most of the math we study in school is very abstract, with no obvious connection with daily life. Further, we often let math become an obstacle to our activities. If you ever hear yourself saying, "I'd like to but I just don't have the math" stop right there. A lot of the work in worthwhile activities is boring (folding 1000 paper birds?) but we don't let that stop us. Get the math. It's not really that hard.

Even elite mathematicians come across things they don't understand (usually in papers by other elite mathematicians) and need review and study to make them clear. Math class did not make these people elite mathematicians, they just figured out how to study and learn on their own.

OK, we all have limits. The first step to befriending math is to find out where our personal limits are. The following is a sort of guide (like those charts of birds or fish) to math. It's not a complete description (I have one that is over 1000 pages long) but it will identify the most common areas and point out how they are used in arts computer programming. Browse through these descriptions and determine where your comfort level ends.

Types of Numbers

Integers 0 1 2 3 4 5 4 - 2 etc.

Integers are used for counting things. This is the one kind of math we can't do without, and we spent a lot of our childhood getting a grip on them. The concept of counting can get quite complicated, because we often count groups of things (like dozens of eggs) then multiply to find out how many that is. (We do that because it is faster to count the boxes than to open the boxes and count each egg.) We often divide our groups into several smaller groups, and want to know how big the shares will be ($17/5$) and how many are left over ($17\%5$). Even with integers, the simplest math, we run into the worst feature of math, which is inconsistent and conflicting use of symbols. For instance, a multiplication may be written 3×3 , $3 * 3$, or $3 \cdot 3$. Those symbols represent different things in other types of math.

Zero is a very important integer, as it represents nothing to count. Negative integers represent the lack of something.

Floating Point Numbers 0.0 1.5 -23.785674

Floating point numbers are also called decimals, although that is not quite accurate (there can be binary floating point numbers). Floats are used to measure things, another unavoidable part of life. When we make a measurement like length, we are comparing what we have to some standard size so we can tell someone else how big it is or if it will fit with something else we have. Integers don't work for this because things are seldom an exact multiple of the standard. We need a fraction to make the measurement useful and get numbers like $6 \frac{7}{8}$ ths or 6.875. We begin to raise a sweat when we divide up measurements, or try to convert measurements made in feet to meters¹. When we don't know what the actual size of something will be (such as the size of a projection screen) we assume the biggest size we can have is 1.0. This is called normalization, and actually makes the math easier.

The normal use of 0 in floating point math is to represent the starting point. Negative floats indicate the measurement was in the opposite direction.

¹ I can never remember the conversion factors and I don't try. I just make all measurements in one system or the other, even if I have to measure things twice.

Addresses 1306 1st st. array[0,0]

An address is a use of numbers we usually take for granted. When we look for 514 Front street and see a door marked 410, we know which way to turn without realizing we are doing math. There is usually some system to the addresses in a city, and we soon figure out what it is. Phone numbers are addresses too-- they are a map to the switching steps required to connect a pair of phones. In computer programming, addresses are numbers that point out the location of other numbers, and we often do math on those addresses just like we do math to find our way around town. We won't deal with actual memory addresses, but we often specify a location as some number of steps past a named location.

Addresses are usually integers (a fraction will put you between two houses) but we sometimes use floats for addressing. There are two reasons to do this. When we deal with things of mixed sizes and resolutions it's easier to define the size as 1.0 and let the program sort out the actual addresses. When we enlarge images and similar data, we need to make up values that would be between the actual pixels. When the program computes a final address for a value lookup, a negative result will usually be an error.

Addresses in towns seldom start with 0, but addresses in computers always do.

Coordinates 0,0 36,24 339,219

A lot of the world of computer graphics is two dimensional, so we find things via a pair of numbers called coordinates. These are measurements from some arbitrary starting point called the origin. The upper left corner and the center of the screen are two popular origins. The two numbers of the coordinate are often called x and y, a reference to the use of letters in algebra. We handle 3D graphics by adding a third number, z, to the coordinate system. Coordinates can be negative numbers-- a negative z means away from the viewer. This style of grid coordinates is called Cartesian, after the mathematician Descartes. Coordinates which have 0 y values run along a horizontal line called the X axis. Coordinates with an x value of 0 are on the Y axis.

Modular Arithmetic 12:00 180°

Things that come in repeating cycles, like days of the week, are usually numbered with a small group of numbers that are reused each time around. For instance, the hours of the day can be numbered from 0 to 23. If we look ahead 25 hours, are referring to an hour after this time tomorrow. This kind of math is often called modulo or modulus math. The most common modular system deals with circles, which are divided into 360 parts for reasons best known to the ancient Babylonians. You can go 480 degrees around a circle, you then subtract 360 to find out where you are. Degrees actually refers to the angle of a slice taken starting from the center of a circle. Eventually someone noticed that doing math with circles was a lot easier if you thought in terms of the radius, so we started dividing circles and angles into 2π radians. The study of circles leads quickly to trigonometry, which I will discuss shortly.

Polar Coordinates 21mi 170°

Mariners and pilots have to navigate over areas that are not nicely cut up into grids like Salt Lake City. They go direct to a destination defined by a compass heading and a distance. Compasses are round, and marked in degrees. North is 0° (or 360°), east is 90°, and so forth. Some kinds of graphic operations² are simplified if we use this polar notation to describe locations-- placing things at some distance and angle from the origin. Sometimes we switch our description from Cartesian to Polar systems and back, a complex bit of math that is usually handled by routines built into the programming language. When this conversion is done, 0 degrees (or radians) points right, in the direction of the X axis, and the angle increases as you turn counter-clockwise³.

Vectors $\vec{A} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

Vectors are numbers that represent quantities that have direction and magnitude. It takes at least two numbers to represent a vector, more if the starting point has to be defined. In the simplest case, the vector [2 3] may indicate draw a line right 2 and up 3 steps. Vectors are associated with coordinate systems, and can be polar or rectangular. We use them in drawing and modeling motion.

Binary Numbers 01010110

The numbers we use most of the time are decimal numbers, meaning we count by 10s. "121" means one hundred plus two tens plus one. Numbers can have other bases, which changes the meaning of "10" but not much else. In particular, binary numbers (which are counted by 2s) are used in computing systems because the circuits are simple. You may also see hexadecimal numbers, which count by 16s, using A B C D E F as additional digits. Some programmers favor them because they reflect the 8 bit underlying architecture of computer memory. Computer languages use some sort of prefix to indicate non decimal numbers, like 0x1A, but each language has a different scheme. The only use of different number base I have seen lately is to define colors, and that is not required.

Imaginary and Irrational Numbers

There is no such thing as the square root of -1. The rule of multiplication says that if the signs of two numbers are the same, the product is positive. So what number can be multiplied by itself and produce -1? However, there are some things that would be useful if we could only square something and get a negative, so mathematicians made one up. They call it i, for imaginary⁴, and use quantities like 0.3i for exotic operations like the Fourier transform. You are unlikely to use imaginary numbers, but they can make some pretty pictures such as the Mandelbrot set. If you see i in a formula, it is not necessarily $\sqrt{-1}$. It may be just a variable.

² Many kinds of reflection, for instance.

³ Unlike compass headings, which increase the other way.

⁴ Mathematicians call it i. Electronics engineers call it j. Go figure.

Irrational numbers exist, but cannot be accurately written down. We don't worry about it, we just use 3.14159 as an approximation of π .

Boolean Numbers

There are only two Boolean numbers, 0 and 1. These are also often referred to as false and true. This is not so much math as shorthand for certain styles of logical thinking. It is systematic and consistent, so computers can follow rules stated this way.

Operators

Numbers are the raw materials of math and operators are the tools. Operators are one source of confusion for ordinary students. All fields use the same set of symbols, but sometimes in different ways. Take the dot in $a \cdot b$. In physics, that would just be the product of a and b , but in vector algebra, that means $A_x \cdot B_x + A_y \cdot B_y$. Sometimes when reading articles in math journals, you discover the authors making up a symbology that can include anything on the typewriter, plenty of Greek, German and Hebrew letters, and things that look like kid's squiggles. The next article will use similar symbols in a completely different way. Here is a field guide to math operators:

Addition and Subtraction

$2 + 2$ $3 - 1$

The dash by itself in front of a number is negation, which might be defined as subtracting from 0, but mathematicians consider negative numbers to be a specific type which has certain restrictions. Note that the order of addition is not important, but the order in subtraction is. In Max, we see the odd notation $! - 3$. This means the operands are swapped, so $1 ! - 3$ is 2. Max is the only place I have seen this.

Multiplication

$a * b$ $a \times b$ ab $2(a+b)$ $a \cdot b$

$3 * 2$ is just shorthand for $2+2+2$.

Division

$\frac{1}{2}$ $1/2$ $1\%2$
 2

Division works in different ways for integers and floats. This is why computer languages make you decide ahead of time if variables will hold floats or integers. When you write numbers, the compiler usually decides based on the presence of a decimal point. But some require you to write 2.0 to get a float equivalent of 2. $7 \% 2$ does an integer division, but returns the remainder. Max also has a $!/$ operation which reverses the operands.

Power

2^2 $2 \wedge 2$

2^3 is shorthand for $2 * 2 * 2$. The caret \wedge can also mean exclusive OR in binary logic

Roots

$$\sqrt[2]{3} \quad 3^{\frac{1}{2}}$$

Finding roots is where most of us stop doing math in our heads and pick up a calculator.

Parenthesis

$$() \quad 2(a + b)^2$$

Parentheses are used to organize operations in computer programs- anything inside the parentheses is done and the parenthesis is replaced with the result. In algebra, parentheses define groups, and operations are performed on the whole group. The second statement comes out as $2a^2 + 2ab + 2b^2$.

Summation

$$\sum x_i$$

The big sigma indicates some kind of summation. It is usually followed by a formula that is calculated over and over and the results added together. There may be tiny statements above and below the sigma telling where to start and stop. This kind of notation is used in statistics and calculus. Some composers use these techniques in algorithmic composition.

Vector operators

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} \quad \begin{bmatrix} A_x \\ A_y \end{bmatrix} \cdot \begin{bmatrix} B_x \\ B_y \end{bmatrix} \quad \begin{bmatrix} A_x \\ A_y \end{bmatrix} \otimes \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

Math with vectors is complicated because each vector includes two or three values. The rules for some of the operations are tricky, but they are usually built into the programming language. Vectors are used in controlling motion and drawing commands, but for this all we need to do is add them.

Matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Arrays of numbers are called matrices. The numbers in an array usually come from the coefficients in a set of equations. Matrices may be added subtracted and multiplied, although the rules in some of these operations are complicated. Matrices can be used to modify vectors. In computer graphics, three dimensional objects are transformed by matrix operations. The branch of math that involves matrices is called Linear Algebra.

Logic

$$\&\& \quad \parallel \quad < \quad > \quad <= \quad >= \quad == \quad != \quad !$$

These combinations are used to perform Boolean logic operations. The equal sign is doubled in computer contexts to make it distinct from value assignment, which co-opted the single =.

Bitwise math

$\& \mid \ll \gg + \wedge !$

These combinations are used to perform math on the individual bits of binary numbers. For many decades, these were the only operations available and everything else was based on combinations of these. Bitwise operations are still useful to hard core programmers, but there are usually less obscure ways to get the work done.

Strange relations

$= \equiv \approx \neq \cong$

These, and many variants are used in statements that describe relationships. They mean equal, identical, approximately equal, not equal, approximately equal.

Fields of Mathematics

Books and courses about math are organized by disciplines or field of study. Half of the work in finding the meaning of some formula is discovering what field it comes from so we can use the operators in the right way.

Arithmetic

Arithmetic is what we studied in grades 1 through 7. It's all the common operations with simple numbers that represent quantity and measurements. Extreme arithmetic can lead to accountancy, where you have concepts like negative expenditures and parenthesis indicating subtraction.

Algebra

Most of us went on to algebra after we gave up on arithmetic. It was probably never explained that algebra is a language, not a set of computations. The whole point of algebra is not to solve problems, but to find ways particular problems can be solved. Given a train leaving Cincinnati at a speed of r at the same time a train leaves Pittsburgh, (d miles away) at a speed p , the time until the wreck can be calculated by $t = d/(r+p)$. The point of this is it will apply to any two trains and any two cities. We are done with the algebra and can just do the calculation. The best way to handle algebra is to let other people do it for us. If you have a problem of this type, do a search and you will find a formula. You only have to know enough about algebra to perform (or program) the calculation. Of course about 20% of the formulas on the internet are wrong. You had better find at least 3 solutions and confirm their effectiveness with some simple experiments.

Of course if you understand algebra, you can find a formula close to what you want and modify it to fit your needs. For instance, we can find the distance from Cincinnati to Pittsburgh by $d = t(r+p)$. This sort of thing is explained in the first few chapters of any good algebra book. The best way to choose a book from the dozens available is to look at the beginning and look at the end. If you can understand the beginning, but not the end, it

is a good book for you. You can pick up Practical Algebra: A Self-Teaching Guide, by Peter Selby and Steve Slavin for \$12 at Amazon.

Do not confuse the language of algebra with any computer language. They use the same symbols, but quite differently:

In algebra $F=ma$ is a fact. If you know ma and a , you can find out F . With a little more effort you can find a knowing F and m .

In the Java computer language, $F=m*a$ multiplies m times a and puts the result in a variable named F . (ma would be another variable name.) That's it. There's no way to reinterpret this statement.

In the LISP computer language, that statement is an error. The equivalent is (setq F (* a m)).

Geometry

Most of us encounter geometry after algebra. It's just practical application of algebraic formulas worked out by other folk. The practical aspects of geometry are useful if you build things. You can find out why a 10" pizza cost twice as much as a 7" pizza and is a slightly better deal. Of course geometry is often used to introduce the concept of logical rigor, of systematically starting with simple postulates and defining more complex statements. It's another case of the trip being more important than the destination. Perhaps if you can prove the sum of angles in a triangle is 180° , you won't believe some lawyer in Nigeria needs your bank account to move his fortune out of the country. Axiomatic thinking may not be everyone's cup of tea. The net has all of the formulas you will ever need, and those are generally correct.

Extreme geometry is topology, which is about how areas can surround and overlap each other without worrying about their actual shape.

Trigonometry

Trigonometry is about circles and angles. Since most motion is circular or angular, you need trigonometry to work with objects that move. The heart of trigonometry is the motion of a hand as it turns a crank. This is described by two functions called sine and cosine.

Digression: a function is a list of numbers that depend on the value of some given number. These numbers often describe some complex curve-- the sine function is one of the simpler curves. We don't need to generate these numbers, we look them up in the back of a book or get them from memory in a calculator. A lot of mathematics is about looking at a list of numbers and trying to describe a simple formula to generate them, but we'll leave that for the professionals.

These turn out to be the ratio of the height of the hand to the radius of the crank (sine), or the ratio of the motion sideways to the radius (cosine)⁵. Since you would get pretty much the same system by laying the mechanism on its side, the two functions are the same shape⁶. By combining these two functions in various ways, you can describe almost any curve. The functions need angles as a basis, so we often derive an angle from time. This gives us motion. You can learn all of the trig you will need from Master Math: Trigonometry (Master Math Series) by Deborah Anne Ross. In fact there's much more than you need in here, so don't worry if you bail around chapter 11.

Calculus

Calculus is algebra for numbers that don't sit still. With the two trains, figuring out the distance given time and speed is not hard, but what if the train is speeding up or slowing down? Solving that requires calculus. Calculus is also good for geometry problems involving curvy things. The formula $A = \pi r^2$ was derived using calculus. There are two branches of calculus, differential and integral.

In differential calculus, the basic approach is to imagine changing the input variable(s) by tiny steps. The result is a curve that shows how the result changes, known as the derivative. The function you are looking at is written $f(x)$ and the derivative function is written $f'(x)$. If you solve the derivative function for a particular x , you get the slope or rate of change of the function at x .⁷ So, if $f(x) = x^2$, $f'(x) = 2x$. Differential equations are usually easy to convert to computer algorithms-- the Lorentz butterfly curve is a system of three differential equations.

In integral calculus we are usually doing the opposite, trying to derive a basic function from a description of how the value changes⁸. The theoretical foundation for this is finding the area between a curve and the axis. This is done by dividing the area up into smaller and smaller rectangular segments. As the rectangles get smaller, their sum approaches the actual area, which is known as a limit. This process is indicated by a stretched out s known as the integral sign, so the math looks like:

$$\int 2x \, dx = x^2 + C.$$

There is also a dx to indicate what part of the formula is being changed. The C represents an offset of the function which is unknowable.

⁵ The ratio of height to sideways is the tangent, an occasionally useful but mostly uninteresting function.

⁶ But have different relationships to the crank angle.

⁷ An alternate notation uses dy/dx instead of $f'(x)$.

⁸ Which is a function, since a function is really just a list of numbers. The equation that describes a function, the graph of the function and the list of numbers are all called the function..

Calculus is like algebra, a way of finding solutions to problems, not solving problems. Most of the common functions have been solved, and we look up the answers in a table of derivatives or a table of integrals. The above formula tells us that if we want a value that doubles on each step, we should square the input.

Calculus is used practically everywhere in science and engineering, but is seldom required in art. The curves the functions defined are often lovely, and can form the basis for things like electric sheep.

And More.....

There is probably a branch of mathematics for every mathematician. There are fields like set theory, fuzzy logic, vector calculus, statistics which may occasionally turn out useful for the computer artist, but are hardly required study. Many of our favorite tricks use advanced math like convolution and matrix transforms, but if you need to write them into your code, you can get them from sources like graphics gems. (www.graphicsgems.org)

So, how do you make mathematics your friend?

First step is an honest self-assessment of your current skill level. Many students sell themselves short, assuming they can never get anything mathematical. Rate yourself in all of the areas listed and decide if those topics are important to you. If you feel the need for improvement, follow these steps:

1. Get a book. Online study is nice, but for math you need to be able to highlight and write in the margins, not to mention stick post-it markers on the pages with vital formulas.
2. Do the exercises in the book. Really do them, with pencil and paper. Don't just read them over and decide they are too easy.
3. Find a website with math exercises and do those.
4. Make up some more exercises and do them. If you change one part of an exercise and compare the result with the answer to the original, you will really get an understanding of what is going on.
5. Put your knowledge to use. Make up problems based on your everyday life. Calculate the area of your desk and volume of your refrigerator.
6. Write programs to solve the problems in the book. After all, we want to learn programming- math is just a step along the way.
7. Work with a friend who is a little bit better than you. Avoid math wizards, they have forgotten what it's like to struggle.
8. Work with a friend who knows less than you. Nothing improves understanding like teaching.

Above all, remember, mathematicians are not born, they are made.