

Trigonometry Notes

Trigonometry is one of those subjects that leaves students asking "What's in it for me?" After all, knowing a lot of stuff about ancient Greeks and how to prove theorems isn't much use to a lawyer or a musician. Graphic arts is another story, at least if you want to write your own programs. There are no theorems to prove, but some of the basic ideas are necessary all the time. This paper covers the concepts you are likely to encounter when doing graphics in Max or Processing.

Cartesian coordinates.

Trigonometry lives on graph paper:

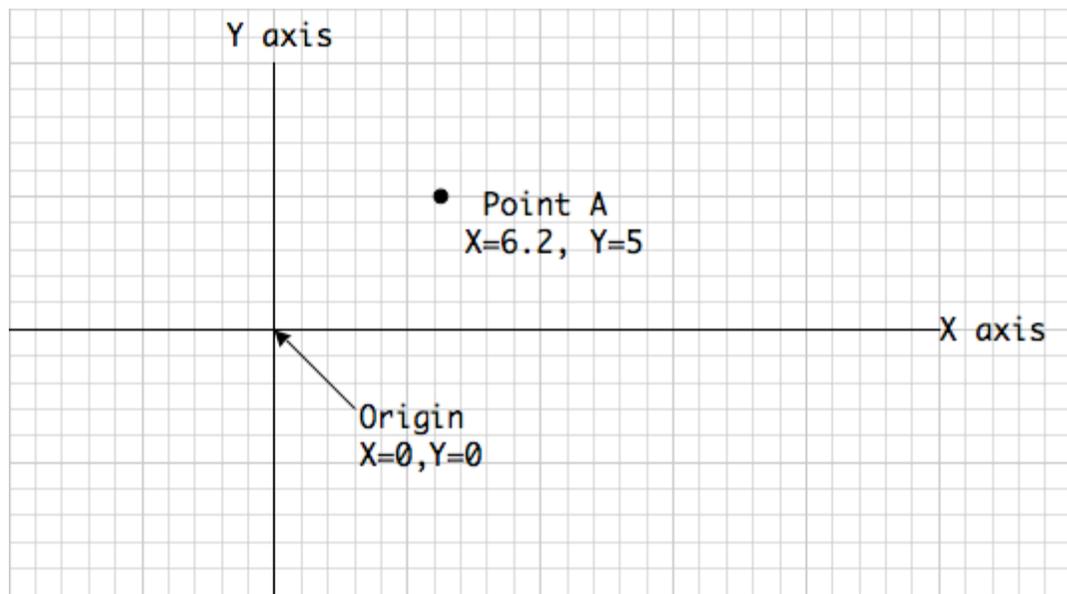


Figure 1. Cartesian space is measured with grid lines.

Figure 1 shows the usual layout. Each line of the grid has a number, and locations on the grid are defined by the intersection of the two nearest grid lines. It's a lot like my saying I live at the corner of Laurel and Cleveland, or I'll meet you at third and Beach. Since this is math, we count to name the lines, and we count starting with 0. The intersection of 0 and 0 is the origin, and lines left of the origin or below it have negative numbers. Also since this is math, we assign letters to indicate horizontal or vertical lines: X for counting to the left and Y for counting up. The horizontal numbered 0 is called the X axis, and the vertical numbered 0 is the Y axis. Any point on the paper can then be defined with an X and Y value. If the point is between lines, it can have fractional values. We always put X before Y, so it makes sense to say point A is at (6.2,5).

Further points about Cartesian graphs:

- There can be three dimensions-- the Z axis points out of the paper, and is counted with positive numbers in front of the page. More dimensions are possible in math, but hard to draw in our universe.
- You are free to move the origin around, as long as you adjust all of the other numbers to compensate.
- This system is named after Rene Descartes, who lived in the early 17th century. He didn't invent it, but his writings about it are so important that other mathematicians copied his notation.

The application of this to computer graphics should be obvious: it's how we describe the pixels on a screen. The origin is the upper left corner, and pixels are counted down and left. Since Y is positive going up and pixel addresses are positive going down, graphics have to be turned upside down before they are displayed. Some programs do this for you, some do not.

Polar Coordinates

There is a second way to look at location. In polar notation, a point is defined by a radius and angle, as if it were on a circle:

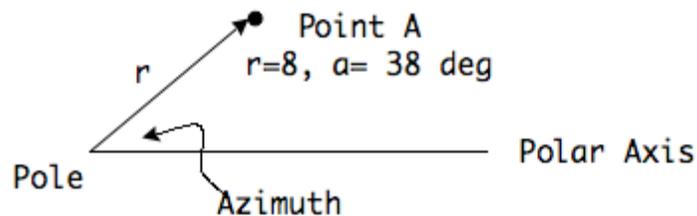


Figure 2.

The center of the circle is called the pole, and is equivalent to the origin in the Cartesian system. (everything is measured from here). A horizontal line from the pole to the right is called the polar axis. To locate a point, draw a line from the pole. The length of this line is the radius r , and the angle the line makes with the polar axis is the azimuth, a ¹. Angles may be measured in degrees or radians. These are both defined in terms of how many are in a complete circle: 360 degrees or 2π radians.²

This may seem like an awkward way to describe locations, but polar notation makes some operations very easy. For instance, rotation around a point just requires a change to the angle. We usually work in Cartesian coordinates, but occasionally convert to polar for

¹ Mathematicians generally use Greek letters for azimuth and angles in general. The most common is θ (theta). My graphic program doesn't do Greek, and Greek in Word doesn't print reliably, so I use roman letters.

² I thought radians were awful until I started doing math with them. Any formula that includes both π and angles in radians is easy to work out because the π terms will cancel out.

some operation, and immediately convert back. To see how to convert, we must revisit the bane of all trig students, the Trigonometric Functions.

Trigonometric Functions

Trigonometry originated in surveying, looking at problems like determining the height of a tree without cutting it down. Many of the problems are about angles and distance, and can be solved by the basic functions of sine, cosine or tangent.

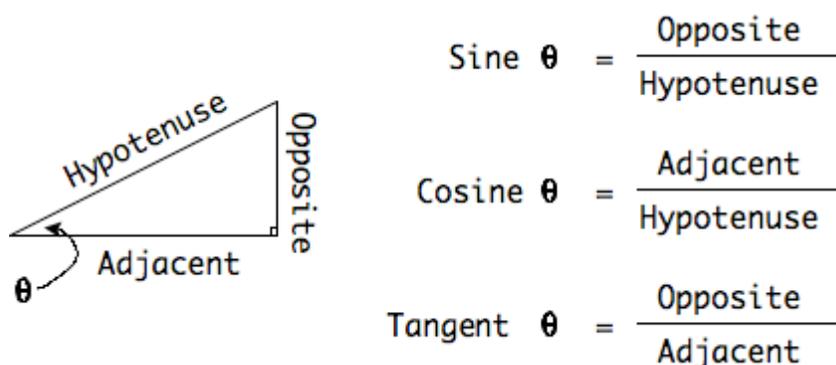


Figure 3.

As figure 3 shows, these functions are defined via a triangle with a 90° corner, or right triangle. The sides are identified as to whether they are opposite or adjacent to the angle we know. (The side opposite the right angle is the hypotenuse.)

For any angle, the sine of the angle is defined as the ratio of the length of the opposite side to the length of the hypotenuse- since the hypotenuse is always the longest side, sines will be a fraction. The sine of a particular angle is always the same- to find what it is, you can just look it up in a table.³

The cosine of an angle is the length of the adjacent side over the hypotenuse. If you look at the drawing, you can see that the cosine of the angle θ is the sine of the unmarked angle, and vice versa.

The tangent is the ratio of the opposite to adjacent sides. Most of the time, we know this ratio, and want to find the angle, so we look for the tangent in the table. This process is called finding an arctangent.

There are other functions, such as secant (hypotenuse/adjacent) but they are aren't used as often. The big three are all we really need to memorize. Among other things, they will allow us to convert between polar and Cartesian coordinates.

³ Many mathematicians spent their careers constructing tables of sines.

Cartesian to Polar and Back

Figure 4 shows how Cartesian and Polar coordinates are related.

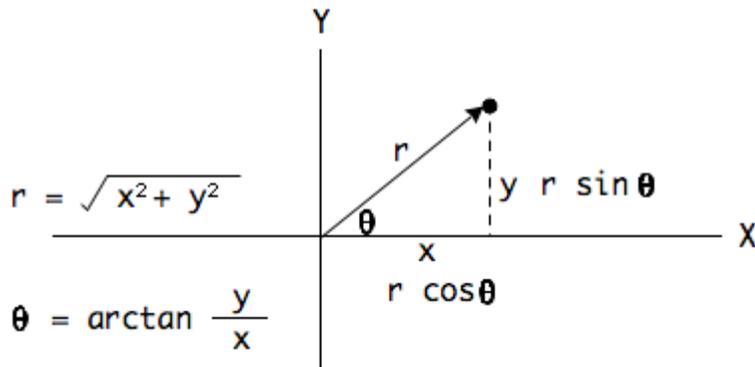


Figure 4.

To go from Cartesian to Polar, the radius is the square root of the sum of the squares of the other two sides. The angle is the arctangent of the ratio of opposite to adjacent. In C or Java code that would be:

```
r = sqrt(x*x + y*y);  
theta = atan(y/x);
```

Of course in Max the whole thing is calculated in cartopol. The angle is reported in radians. To go back after the process:

```
y = r*sin(theta);  
x = r*cos(theta);
```

Or the poltocar object. One other interesting use of these formulas is that if you plot them with a steady radius and a theta that hits most of the numbers between 0 and 2π , you will get a circle.

Plotting Sine Functions

We don't really need to bother ourselves with the numbers in the sine table, the computer looks them up for us. However I often do play with these numbers, because they are beautiful. Figure 5 shows a few examples of sine based drawing. In the later cases, lines are drawn between points on two different sine waves.

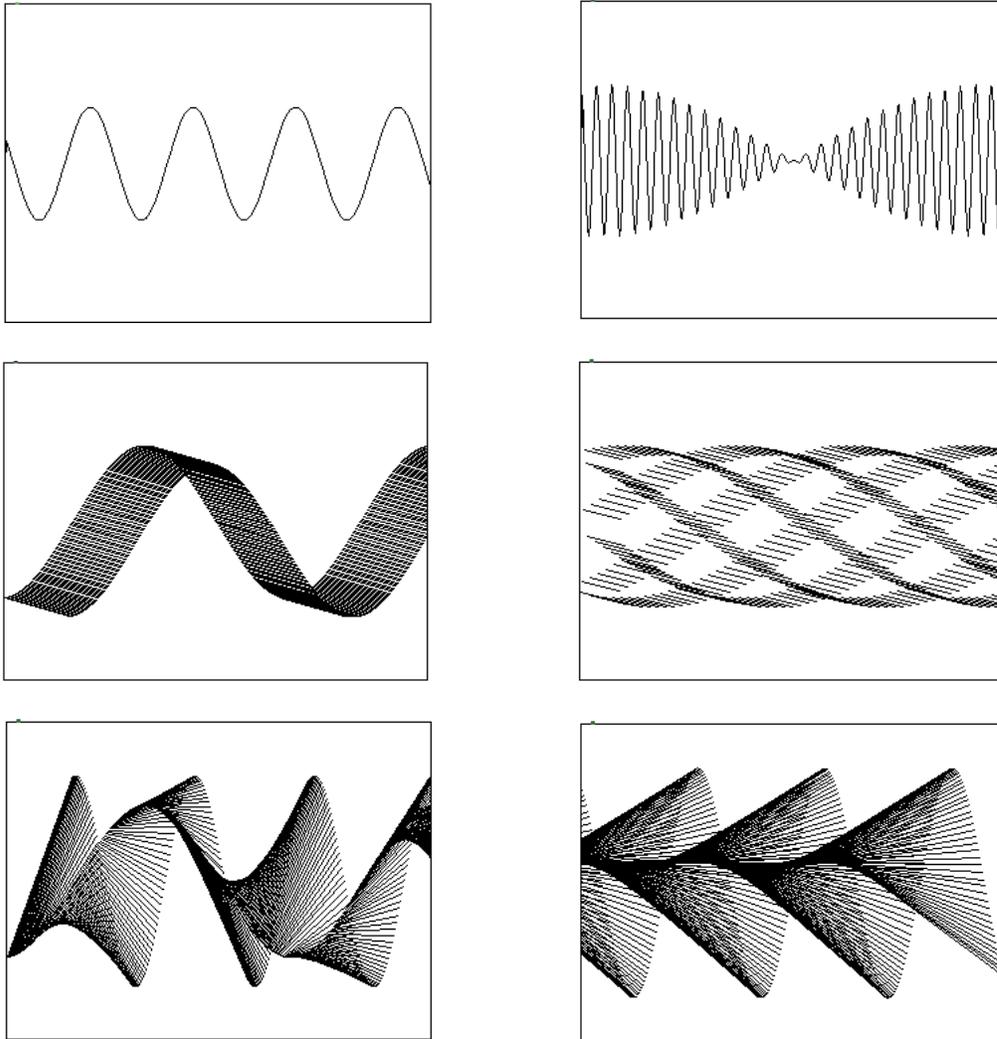


Figure 5. Sine wave art. These patterns often move slowly across the screen.

As I mentioned before, plotting $x=\cos(\theta)$ and $y = \sin(\theta)$ will result in circles.

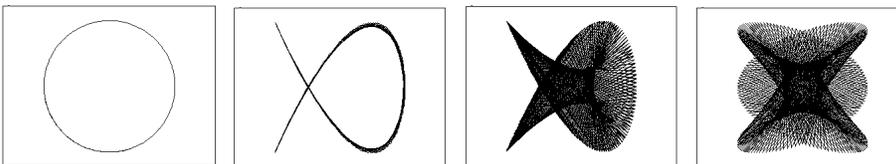


Figure 6. Lissajous patterns

If you use different angles in the sine and cosine terms you can generate many complex forms based on Lissajous patterns (See my tutorial; Lissajous art).

If you start with a polar view of things you can make spirals and flowers by manipulating the radius. The images in figure 7 are based on that technique.

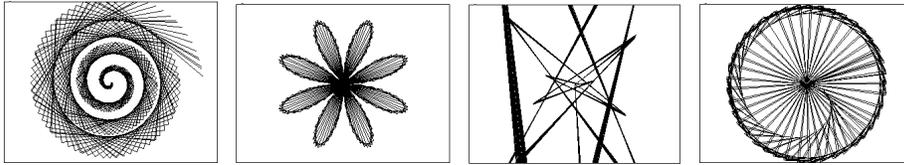


Figure 7. Polar images. These can be made to spin.