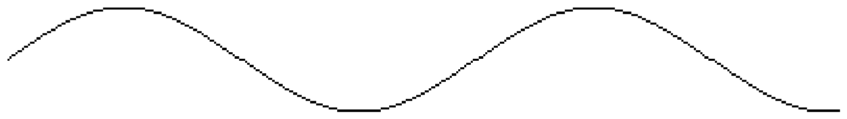
Notes on Fourier Transforms

The Fourier transform is something we all toss around like we understand it, but it is often discussed in an offhand way that leads to confusion for those just learning their way around DSP. I'm not ready to write a comprehensive manual on the thing (Smith devotes 14 chapters to it), but here are some assorted bits of trivia that may clear the air some.

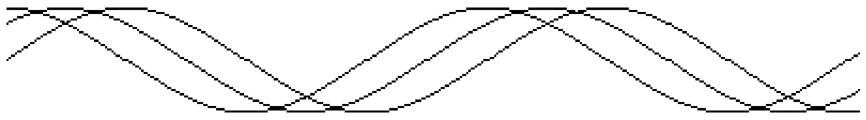
The Fourier transform is about graphs and curves. It is useful in sound analysis, but it can be applied to other fields as well. When Jean Baptiste Joseph Fourier invented it in 1807, he was investigating the propagation of heat in metals.

Reminder about trigonometry

A sine function is a graph of the type Y = sin(X).

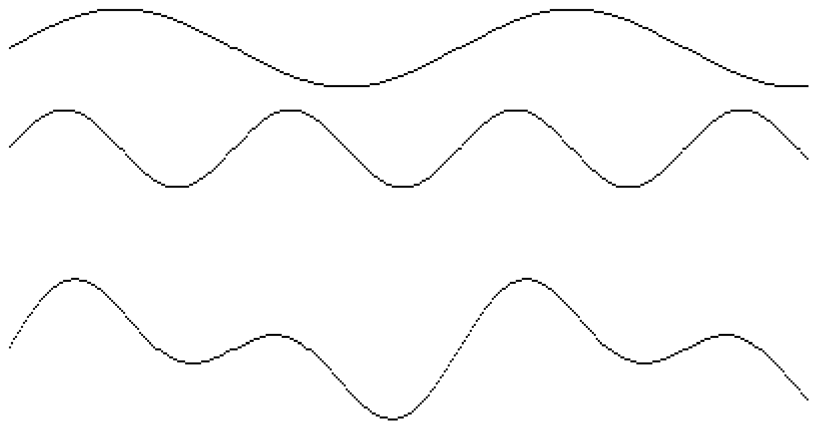


This curve starts with Y=0. If it started with another value, it would be Y = sin(X +p) with p indicating phase. A sine curve of some given phase can also be represented by Y= A sin(X) + B cos(X). Here are some curves with the coefficients A= 1 B=0; A=0, B = 1; and A= 0.7, B= 0.7. Can you figure out which is which?

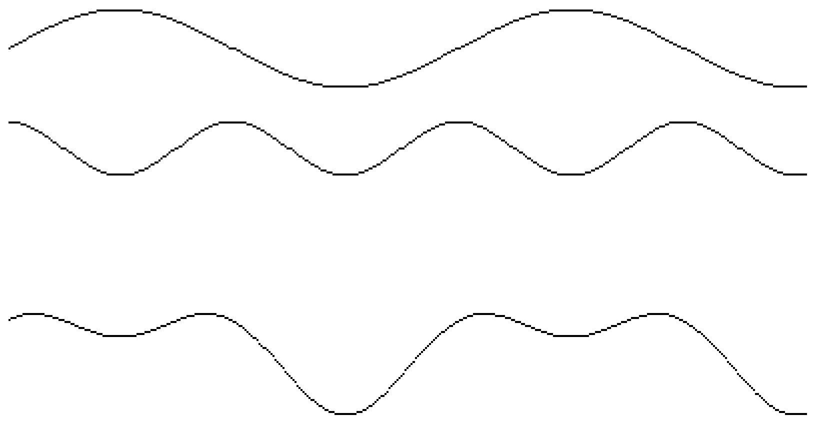


Hint: sin(0) = 0 and cos(0) = 1.

Here's an example of adding two sine functions
Y = A sin(X) + B cos(X) + C sin(2X) + D cos(2X)



In this case, A and C are 1 and B and D are 0. Now the same exercise with C = 0 and D = 0.7



The resultant curve looks quite different. You can build up quite complicated curves by adding sine and cosine functions at more frequencies. If the frequencies used are harmonically related, the resultant will be a repeating waveform.

Fourier Analysis is about taking complex curves apart.

Buzzwords, or fftspeak:

Time domain representation means a graph with time along the bottom. Waveforms are usually represented this way.

Frequency domain representation means a graph with frequency along the bottom. Spectral plots are like this. The domain runs from 0 hz to the sampling frequency.

Polar representation means graphing in terms of an angle and radius. Since sine waves are inherently angular this can be useful. To map the frequency domain onto a polar plot, the angle π radians represents 1/2 the sampling rate. The region on the bottom of the circle represents negative frequency plotted from 0 to -π. Points on a polar plot can also be indicated by their rectangular (Cartesian) coordinates.

A **function** is the mathematical representation of any sort of curve. A sine wave function could be written $f(t) = k\sin(\omega t)$. Functions are always functions of some thing, in this case t. Some texts make the distinction that f(x) is a continuous function (i.e. an unbroken curve) and f[x] is a discreet function, made up of a lot of points. (Like a sampled waveform.) f(x) is pronounced "f of x".

Terms of a function are the parts that are separated by + signs.

A **coefficient** is a value that adjusts the overall value of a term in a function. For $k\sin(\omega t)$, k is a coefficient. Most of the hard part of DSP design is finding the coefficients that make a function give a desired result.

The **unit** means one. The unit circle on a polar plot is a circle of radius 1. Setting things to equal 1 often makes math clearer.

The **delta function** is a 1 followed by as many 0s as you want. It's the mathematical equivalent of hitting something with a stick to see what it will do. When you apply a delta function to digital filter, you get its **impulse response**. The Fourier transform of the impulse response is the frequency response.

The letter **j** is the imaginary square root of -1. Some mathematicians use i for this, but i means something else in electronics. We aren't interested in roots of negative numbers per se, but complex numbers are handy.

An **imaginary** number is one that has j (or i) as a factor. 7j is imaginary. It Follows that 7 is a **real** number.

Complex numbers are the sum of a real and imaginary part such as (a + bj). The math works as if the imaginary part were at right angles to the real part, which is the way a lot of audio phenomena behave. Some authors indicate variables that represent complex numbers with a capital letter. Numbers whose imaginary parts are zero are real numbers. Complex numbers may be represented by the real and imaginary coefficients or in a polar form.[1]

[1] Either way, they map onto a grid with real across the X axis and imaginary on the Y axis.

The letter **ω** (Greek lower case omega) is often used to refer to angles. You will see $\omega=2\pi f$ as a way to convert a frequency f to an angle. When derived this way, ω may be called angular frequency. You will often see ωt in the literature, which simply means we are plotting some function of ω at time t.

The letter **e** means Euler's constant, a number like π that is one of the fundamental features of the universe. It is used to calculate interest, and is the base of natural logarithms. There is a famous formula called Euler's relation, which proves $e^{j\omega} = \cos(\omega) + j\sin(\omega)$. So you can represent a sine wave as $e^{j\omega}$.

You **multiply** two **functions** by multiplying the value of each point on one curve by the value of the equivalent point of the other curve.

Correlation of functions is performed by multiplying each point on one curve by all of the other curve. This gives one complete curve per point. You then add all of these together.

Convolution of functions is performed by multiplying each point on one curve by the reverse of the other curve and adding all the results. Convolution of two time domain functions is equivalent to multiplying the frequency domain versions, and vice versa.

A **transform** is a method for converting a function of time into a function of frequency (or back). In audio, transforms convert waveforms into a spectral representation or back.

Transforms

There are several transforms out there - Laplace, Z-transform, and Fourier being the big names.

The **Laplace** transform converts a waveform into a series of exponentially decaying sinusoids. This results in a whole family of curves of amplitude vs. frequency (one curve for each possible exponent) These are represented by parallel curves in a three dimensional space called the s domain. This is very useful for designing analog filters, whose response is a combination of exponential shapes.

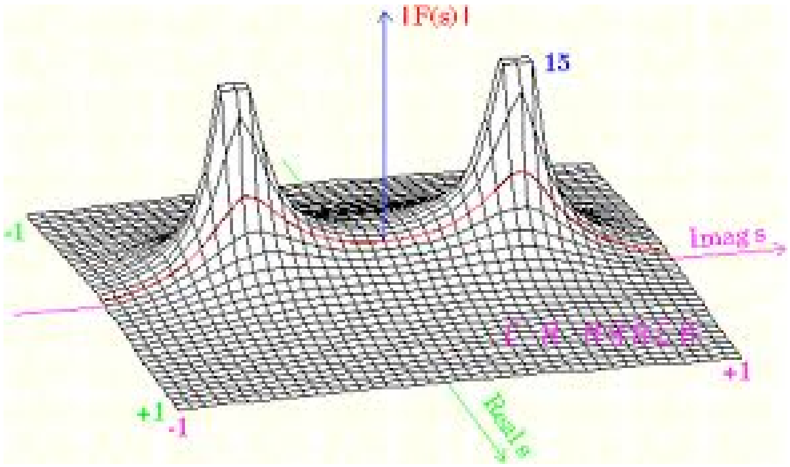


Figure 1. Laplace transform. The curves along the real gridlines are components.

The **Z transform** converts waveforms into something similar to the s plane, but in a polar scheme known as the Z plane. The frequency is represented by the angle, and the exponent by the radius, so the amplitude vs. frequency curves are wrapped in circles. This is used for designing digital filters.

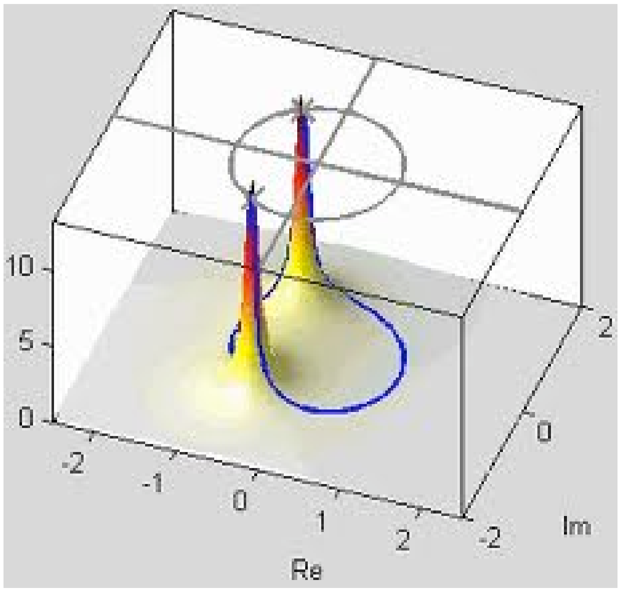


Figure 2. Z-transform. The components are on concentric rings.

The **Fourier transform** converts waveforms into a series of sinusoids. The Fourier transform gives sine and cosine coefficients for all of the component frequencies, so these are plotted in the frequency domain.

Types of Fourier transform

There are four types of signal encountered in audio. These signal types are:

- Non-periodic analog signal

- Periodic analog signal
- Non-periodic digitized (discrete) signal
- Periodic digitized (discrete) signal

Different variants of the Fourier transform are appropriate to each. Some of these are referred to with initials.

- In the case of the non-periodic analog signal, the sinusoids may have any frequency, and there may be an infinite number of them. The original Fourier transform deals with this.
- In the case of the periodic analog signal, the sinusoids take frequencies that are multiples of the fundamental established by the period (The **Fourier series**). There still may be an infinite number. (You need an infinite number of sinusoids to represent a corner in a waveform, so square waves and triangles have infinite Fourier representations.)
- In the case of the non-periodic digitized signal, the sinusoids are discrete themselves, and are limited in frequency to the range of one half of the sampling rate. The **D**iscrete **T**ime **F**ourier **T**ransform (DTFT) is used here.
- In the case of the periodic digitized signal, the sinusoids are discrete themselves, and limited in frequency to a harmonic series up to one half of the sampling rate. This can be analyzed by the **D**iscrete **F**ourier **T**ransform, (DFT) or the **F**ast **F**ourier **T**ransform (FFT).

All of these transforms have inverse transforms, which take us back to the original waveform.

The transforms for analog signals are the theoretical basis for all of the math, but you can't actually do any of them in a computer. Computers deal with discrete signals because all they know is lists of numbers. These are processed by the DFT. The waveform goes in as a list of numbers, and two lists come out. What numbers do we need in the lists?

Details of the output

A sinusoid has frequency, amplitude and phase. These three parameters must be included for each of the components of a waveform. When we are dealing with periodic waveforms, the frequency of a component can be implied by its position in the list. The frequency positions are called **bins**. The first component is DC, the next the fundamental, then the second harmonic, and so on. So lists with values for each component will be sufficient. What do the two values mean?

Amplitude and phase are one possibility. In that case each pair of numbers is a **polar representation** of the component.[2] This is very nice, because the list of amplitudes can be charted as a graph of the frequency response. The frequencies represented actually range from –sr/2 to sr/2, but we usually only look at the positive side of the graph. The negative side is a mirror image of this.

We generally ignore phase when we are just looking, since phase has no audible effect.[3] For mathematical convenience, the amplitude range is normalized from 0 to 1.0, and the phase is from $-\pi$ to π.[4]

Phase is hard to do math with. The fact that the phase wraps around from $-\pi$ to π, and that math leads to division by 0 makes the code awkward. For computation, a **rectangular representation** of amplitude and phase is handier. This can be done by specifying each component as a sum of a cosine wave and a sine wave. This is the most common definition of the Fourier series:

$$a_0 /2 + (a_1\cos(x) + b_1\sin(x)) + (a_2\cos(2x) + b_2\sin(2x)) + \ldots.$$

This is usually output as two lists, one with the a values and the other with the b values. To keep the lists the same length, a b_0 of value 0 is included in the b list. The lists are called the cosine and sine parts or sometimes the real and imaginary parts. If there are N points in the input sample, there will be $N/2 + 1$ points in each list. The frequency x is the sample rate divided by N. A 512 point DFT with a sample rate of 44.1 khz gives a fundamental frequency of 86.13 hz. The highest frequency represented is half the sample rate. (The a_0 term is a DC offset)

This is called the real DFT. The algorithm that does this is rather inefficient (there's a correlation with every potential sinusoid), so a streamlined version called the **F**ast **F**ourier **T**ransform is preferred. The FFT also outputs two lists, , which are the coefficients for the real and imaginary parts of complex numbers.

$$(a_0\cos(0) - b_0 j\sin(0)) + (a_1\cos(x) - b_1 j\sin(x)) +(a_2\cos(2x) - b_2 j\sin(2x))$$

These lists can be changed to amplitude and phase pairs with a simple Cartesian to polar conversion.

There are N components in each list from the FFT . They still represent multiples of $x = SR/N$. The values beyond half the sampling rate represent negative frequency components running from $-(N/2 -1)x$ to $-x$.[5]

[2] Phase is an angle, and a radius and angle define a point on a polar plot.
[3] The inaudible effects of phase are significant, and forgetting about phase when you are processing audio and not just listening is a serious mistake.
[4] We usually do angles in radians. It's nicer for the computers. And yes, negative phase is as likely as positive.

The **FFT** takes complex numbers as its input[6]. If the input is a not a complex signal, the imaginary parts of the input are zero and the output curves will be symmetrical about the N/2 point. This implies a lot of wasted computation. This can be skipped, giving the **real FFT**. The output looks just like the complex FFT.

At this point, many may be worrying about how waveforms that are harmonic series on other fundamentals can be accurately represented by components based on the arbitrary frequency SR/N. Consider a 100 hz tone. As an example This falls between the 86.13 hz and the 166.32 hz components of the transform, so it would show up in both bins. These, plus the phase information are enough for the iFFT to give a 100 hz tone back. Of course, the more points in the analysis, the more accurate the reconstructed signal.

Here is an fft of a 229 hz sine wave showing how a combination of real and imaginary bins represents a sine wave of arbitrary frequency. This is a 1024 point fft, which gives a bin spacing of 43.0664 hz. The tallest bin is 215.33 hz. In reality, the bins are constantly changing.

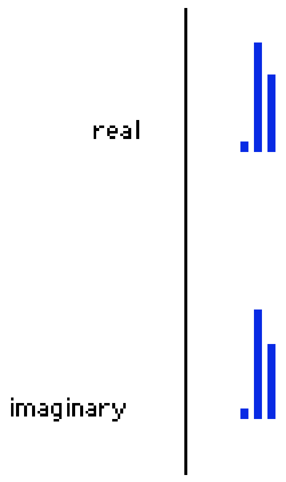


Figure 3.

Transforming continuous signals

The FFT works with a chunk of signal N points long. The algorithm requires that N be a power of two. If a recording is too short, we can just add zeros, but more likely, we are interested in recordings much longer than N samples. We are also interested in how the transform changes over time, not just the overall average frequency content.

[5] This makes the outputs symmetrical curves. For many purposes the negative frequencies can be ignored, but the iFFT uses them when converting back to waveforms, so ignoring them would result in a loss of amplitude.

[6] Most waveforms are real, not complex. Where would you find a complex waveform? As the output of an inverse FFT.

This problem is overcome with a system of windowing, which is similar to the practice of showing moving pictures by projecting a series of still pictures. In essence, the incoming signal is broken into chunks of N points and analyzed as a series of frames. To smooth out the errors caused by this arbitrary chopping, overlapping chunks are processed-- when the signal is reconstituted, the overlapped frames are mixed together. For further smoothing, the frames are faded in and out before the analysis. There are many schemes for doing this. Luckily for Max users, all of this is hidden in the pfft~ object.

Putting the FFT to work.

So, whats the point? Well, there are several pretty neat tricks we can do with Fourier transforms of signals. In MSP the output of fft~ is a pair of synchronized signals, one with the coefficients of the real terms and the other with the coefficients of the imaginary terms. These can be routed and processed just like any other signal. (But you wouldn't want to listen to them!)

Viewing Spectra.

The patch in figure 4 will show the spectrum of a sound

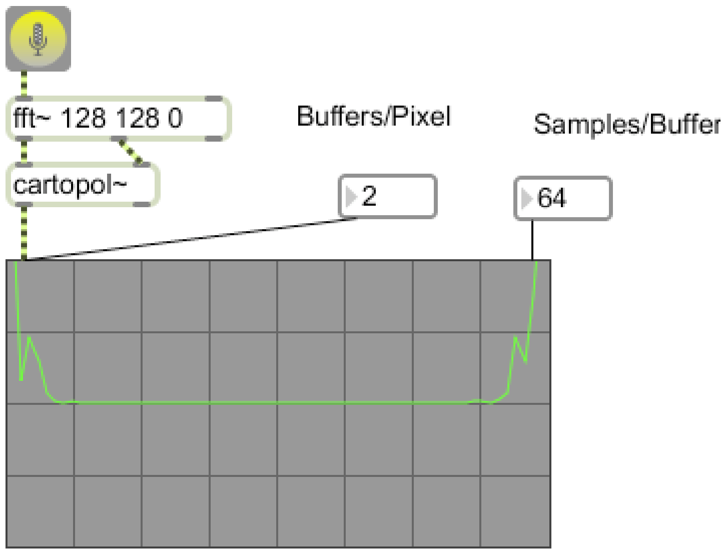


Fig 4. The magnitude spectrum of fft~

Note the magic settings for fft~ and scope~ to get a stable image.

Cartopol~ is used to convert the rectangular output of fft~ into magnitude and phase format. The spectrum is mirrored because fft~ produces the full complex spectrum-- those are the negative frequencies at the right. It's hard to see much detail because of the limitations of scope~. Perhaps the future will see a display optimized for this.

It's not hard to produce a bargraph version of this using a buffer~, peek~ and poke~, as explained in the visualizing music tutorial.

Pitch Shift

First a word about pfft~[7]. This is a wrapper for fft operations. To use it you specify a subpatch to process the fft data, a frame size, and the number of overlap windows to use. Pfft~ will perform the fft and pass the data to the subpatch, and will reconstruct whatever the subpatch gives back. Almost any fft process is going to look like this at the top level:

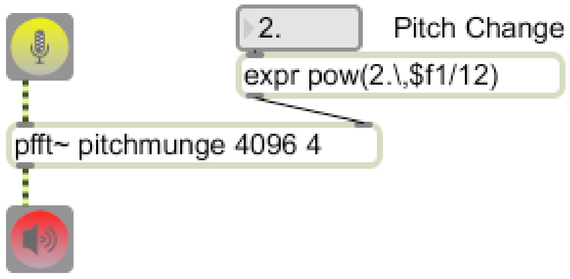


Figure 5.

Pfft~ requires a name for a subpatcher to load in, an fft size, and the number of windows to overlap. A large analysis and many windows give better results and a heavy CPU hit. Generally I get the effect working in a generous setup and then reduce the options until it starts to sound bad. The supatchers look something like this

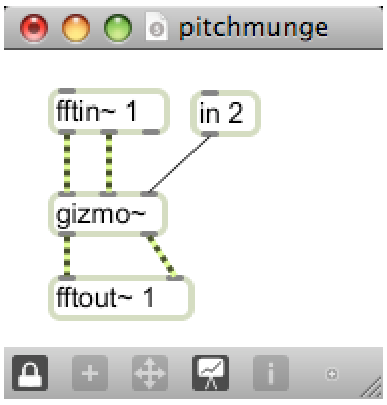
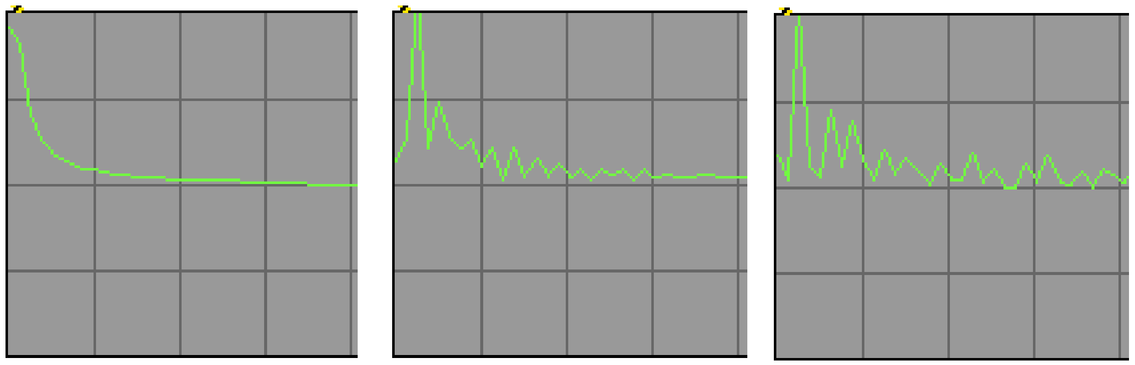


Figure 6.

[7] Poly fft~. It's like the Poly~ object, running multiple copies of the same subpatcher.

In the sub patch, the fft analysis comes from an **fftin~** object. This specifies the inlet number on pfft~ this will be connected to, and a window shape if you like to tweak things. fftin~ has three outlets: one for the real fft signal, one for the imaginary fft signal, and one for synchronization - this output is an index to the bin[8] number in the frame. The inverse fft happens in **fftout~.** The real and imaginary signals applied will be reconstructed at the outlet of pfft~. There is an argument to specify the outlet number.

The patcher shown will use **gizmo~** to shift pitch by semitones. Figure 4 shows what gizmo~ does- peaks in the spectrum are moved by multiplying the frequency to maintain the intervals. (**fbinshift~** shifts all parts of the spectrum equally. This modifies the sound in strange ways.)



No shift 1 octave 2 octaves

Figure 7.

Convolution by multiplication

The convolution of two signals an be performed by multiplying their Fourier transforms. This is done by multiplying the real and imaginary parts of the transforms. The result is a strange amalgamation of the two sounds. Figure 8 illustrates how this is done in the pfft~. It's nothing more complicated than a couple of *~ objects.

[8] The something of the Asomething terms.

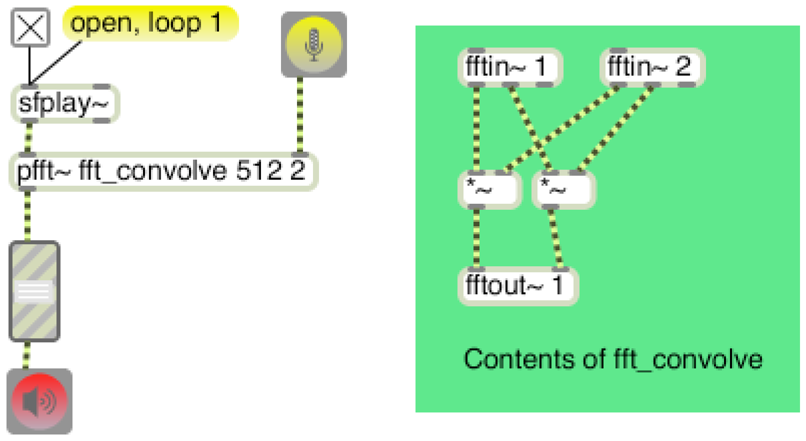


Figure 8.

Filtering by Convolution

When the fft signal has been converted into magnitude and phase format, it should be clear that you can change the amplitude of the reconstructed signal just by changing the magnitude part of the fft signal. You can also do this by changing both the real and imaginary signals in rectangular notation. If you could isolate particular bands in the fft output, you could change just that part of the spectrum. A method for doing this is shown in the demonstration patch "forbidden planet", and deconstructed in figures 9-11.

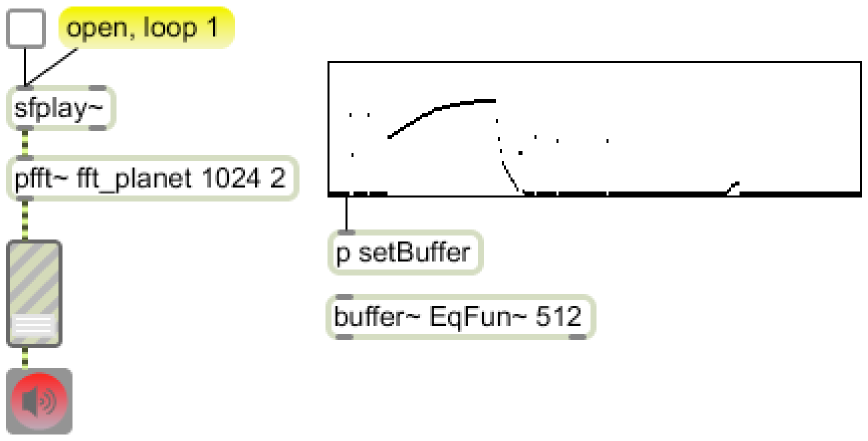


Figure 9. The elements of the forbidden planet patch.

The main patch: a table containing a shaped spectrum is converted into a signal by loading it into a buffer~ called EqFun~. Figure shows the subpatcher that does this conversion.

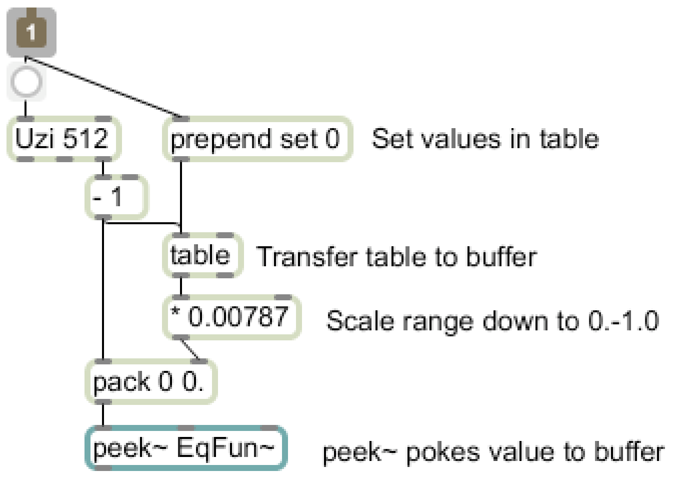


Figure 10. Setting the buffer~

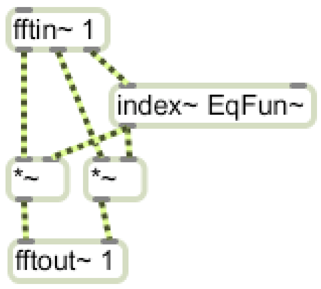


Figure 11. Doing the work

The pfft~ subpatch shown in figure 11. does the processing. The frequency control signal is played in sync with the fft via the **index~** object connected to the third output of fftin~. Both the real and imaginary parts of the fft are multiplied by the control signal and the results sent to fftout~. This will superimpose the shape of the table contents (i.e. the filter curves drawn by the user) on the spectrum of the reconstructed signal.

Vocoding with the fft

The next step is to use an input signal for the modification source. This is illustrated in figure 12.

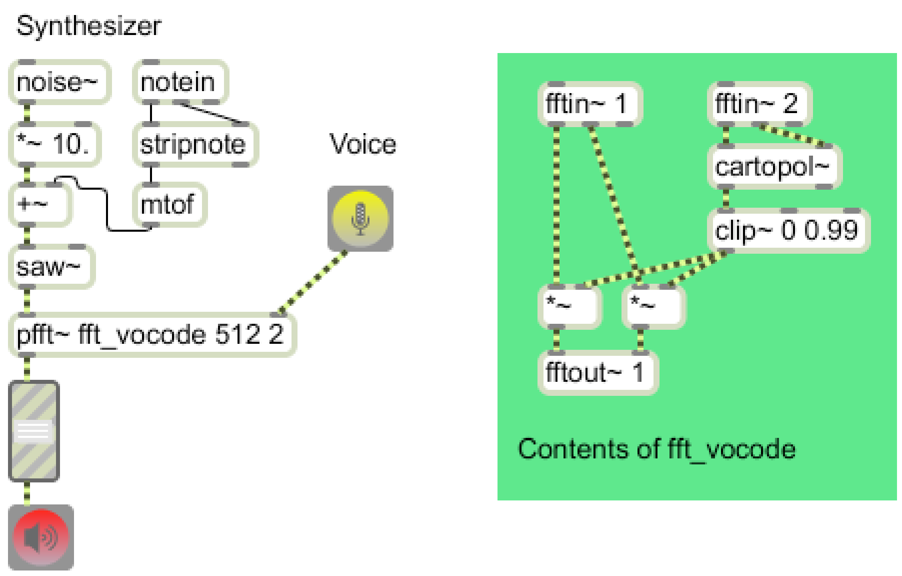


Figure 12.
In the fft_vocode supatcher, the signal from fftin~ 2 is converted to magnitude and phase via cartopol~, and the magnitude is multiplied by the fftin~1 signal. This superimposes the spectrum of input 2 onto input 1. The result is something of both.

In the outer patch, a saw waveform is modulated by noise to smear the harmonics enough to catch matching harmonics of the voice. (You will only get output when the components of the two signals match closely.) This one gives a nice singing robot effect.

Study Further

To learn more about the Fourier transforms, I suggest you brush up on your math, then study one of these:

Loy, Gareth; Musicmathics, volume 2, 2007. The MIT Press
Roads, Curtis; The Computer Music Tutorial 1996. MIT Press
Smith, Steven; The Scientist and Engineer's Guide to Digital Signal Processing, 1997. (Download in pdf or order from www.DSPguide.com)